

# Approximate Eigenstructure of LTV Channels with Compactly Supported Spreading

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**Abstract**— In this article we obtain estimates on the approximate eigenstructure of channels with a spreading function supported only on a set of finite measure  $|U|$ . Because in typical application like wireless communication the spreading function is a random process corresponding to a random Hilbert–Schmidt channel operator  $\mathcal{H}$  we measure this approximation in terms of the ratio of the  $p$ -norm of the deviation from variants of the Weyl symbol calculus to the  $a$ -norm of the spreading function itself. This generalizes recent results obtained for the case  $p = 2$  and  $a = 1$ . We provide a general approach to this topic and consider then operators with  $|U| < \infty$  in more detail. We show the relation to pulse shaping and weighted norms of ambiguity functions. Finally we derive several necessary conditions on  $|U|$ , such that the approximation error is below certain levels.

## I. INTRODUCTION

Optimal signaling through linear time–varying (LTV) channels is a challenging task for future communication systems. For a particular realization of the time–varying channel operator the transmitter and receiver design which avoids crosstalk between different time–frequency slots is related to "eigen–signaling". Eigen–signaling simplifies much the information theoretic treatment of communication in dispersive channels. However, it is well–known that for a whole class of channels such a joint separation of the subchannels can not be achieved. A typical scenario, present for example in wireless communication, is signaling through a random doubly–dispersive channel  $\mathcal{H}$ :

$$r(t) = (\mathcal{H}s)(t) + n(t)$$

From signal processing point of view the preferred design of the transmit signal  $s(t)$  needs knowledge on the true eigenstructure of  $\mathcal{H}$ . This would in principle allow interference–free transmission and simple recovering algorithms of the information from received signal  $r(t)$  degraded by the noise process  $n(t)$ . However, for  $\mathcal{H}$  being random, random eigenstructure has to be expected in general and a joint design of the transmitter and the receiver for an ensemble of channels has to be performed. Nevertheless with such an approach interference can not be avoided and remains in the communication chain. For such interference scenarios it is important to have bounds on the distortion of a particular selected signaling scheme.

First results in this field can be found already in the literature on pseudo–differential operators [1], [2]. More recent results with direct application to time–varying channels were obtained by Kozek [3] and Matz [4] which resemble the notion of underspread channels. They investigated the approximate symbol calculus of pseudo–differential operators in this context

and derived bounds for the  $\ell_2$ –norm of the distortion which follow from the approximate product rule in terms of Weyl symbols. Controlling this approximation intimately scales with the "size" of the spreading of the contributing channel operators. For operators with compactly supported spreading this is  $|U|$  – the size of the spreading support  $U$ . Interestingly this approximation behavior breaks down in their framework at a certain critical size. Channels below this critical size are called in their terminology underspread, otherwise overspread.

Underspreadness of time–varying channels occurs also in the context of channel measurement [6]. See also the recent article [7] for a rigorous treatment of channel identification based on Gabor (Weyl–Heisenberg) frame theory. The authors connect the critical time–frequency sampling density imminent in this theory to the stability of the channel measurement. A relation between these different notions of underspreadness has to be expected but will be out of the scope of this paper.

This article considers the problem of approximate eigenstructure from a different angle, namely investigating the  $\ell_p$ –norm  $E_p$  of the error  $\mathcal{H}s - \lambda r$  for well–known choices of  $\lambda$ . This direct formulation allows for improvements to the existing bounds, generalizations to arbitrary regions of spreading and different distortion measures. Furthermore the approach will show the connection to well–known fidelity criteria related to pulse design [3], [8], [9]. Using the techniques recently presented in [9] we finally extract necessary conditions for  $|U|$  that the error does not exceed certain levels. Furthermore we discover some interesting relations related to underspreadness in form of [3], [4].

The paper is organized as follows. After settling the basic definitions in the first section of the paper the second section reviews the Weyl correspondence and the spreading representation of Hilbert–Schmidt operators. In the next sections we will then consider the problem of controlling  $E_p$  and give necessary conditions on  $|U|$ . In the last part we will verify our framework with some numerical tests.

## A. Some Definitions

But before starting, the following definitions are needed. For  $1 \leq p < \infty$  and  $f : \mathbb{R} \rightarrow \mathbb{C}$  the functional  $\|f\|_p \stackrel{\text{def}}{=} (\int |f(t)|^p dt)^{1/p}$  is then usual notion of the  $p$ –norm ( $dt$  is the Lebesgue measure on  $\mathbb{R}$ ). Furthermore for  $p = \infty$  is  $\|f\|_\infty \stackrel{\text{def}}{=} \text{ess sup } |f(t)|$ . If  $\|f\|_p$  is finite  $f$  is said to be in  $\mathcal{L}_p(\mathbb{R})$ . We will frequently make use of the relation  $\|f^a\|_b^c = \|f\|_{ac}^{ab}$ . The

function  $\chi_U$  will always denote the characteristic function onto the set  $U \subseteq \mathbb{R}^2$ .

## II. DISPLACEMENTS OPERATORS AND AMBIGUITY FUNCTIONS

Time-frequency representations are an important tool in signal analysis and physics. Among them are Woodward's cross ambiguity function and the Wigner distribution. Ambiguity functions can be understood as inner products representations of displacement (or shift) operators, defined by its action on function  $f : \mathbb{R} \rightarrow \mathbb{C}$  as:

$$(\mathbf{S}_\mu f)(x) := e^{i2\pi\mu_2 x} f(x - \mu_1) \quad (1)$$

The functions  $f$  are time-signals, i.e.  $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$  where  $\mu_1$  is time shift and  $\mu_2$  is a frequency shift. However the following can be straightforward extended to multiple dimensions. The operator  $\mathbf{S}_\mu$  acts isometrically on all  $\mathcal{L}_p(\mathbb{R})$ , hence is unitary on the Hilbert space  $\mathcal{L}_2(\mathbb{R})$ . The arbitrariness due to the non-commutativity of shifts in the previous definition can be covered in using generalized displacements of the form:

$$\mathbf{S}_\mu(\alpha) = \mathbf{S}_{(0, \mu_2(\frac{1}{2} + \alpha))} \mathbf{S}_{(\mu_1, 0)} \mathbf{S}_{(0, \mu_2(\frac{1}{2} - \alpha))} \quad (2)$$

We will call  $\alpha$  as polarization. All these operators establish (up to unitary equivalence) unitary representations of the Weyl–Heisenberg group on  $\mathcal{L}_2(\mathbb{R})$  (see for example [2]). In physics it is common to choose the most symmetric case  $\alpha = 0$  and the operators are usually called Weyl operators. The definition (1) appears for  $\alpha = 1/2$  and we call  $\mathbf{S}_\mu = \mathbf{S}_\mu(1/2)$  as *time-frequency shift operator*. If we define the symplectic form as  $\eta(\mu, \nu) := \mu_1\nu_2 - \mu_2\nu_1$ , we have the following well-known *Weyl commutation relation*:

$$\mathbf{S}_\mu(\alpha) \mathbf{S}_\nu(\beta) = e^{-i2\pi\eta(\mu, \nu)} \mathbf{S}_\nu(\beta) \mathbf{S}_\mu(\alpha) \quad (3)$$

In this way the *cross ambiguity function* can be defined as:  $\mathbf{A}_{g\gamma}^{(\alpha)}(\mu) \stackrel{\text{def}}{=} \langle g, \mathbf{S}_\mu(\alpha)\gamma \rangle$ .

## III. WEYL CORRESPONDENCE AND THE SPREADING REPRESENTATION

In the previous section it was motivated that  $\mathbf{A}_{g\gamma}^{(\alpha)}$  yields a local time-frequency description of functions. Now the same can be repeated for Hilbert–Schmidt operators. Let us introduce them as the 2th Schatten class: If we define for linear mappings  $A \in L(\mathcal{L}_2(\mathbb{R}))$  from Hilbert space  $\mathcal{L}_2(\mathbb{R})$  into itself  $|A| := (A^*A)^{1/2}$ , then for  $1 \leq p < \infty$  the functional  $\|A\|_p := \text{Tr}(|A|^p)^{1/p}$  is called the  $p$ th Schatten norm. The set  $\mathcal{T}_p := \{A \in L(\mathcal{L}_2(\mathbb{R})), \|A\|_p < \infty\}$  is called the  $p$ th Schatten class where  $\mathcal{T}_\infty$  is set to be the compact operators. Then  $\mathcal{T}_p$  for  $1 \leq p < \infty$  are Banach spaces and  $\mathcal{T}_1 \subset \mathcal{T}_p \subset \mathcal{T}_\infty$  (see for example [10] or [11]). The sets  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are called trace class and Hilbert–Schmidt operators. Hilbert–Schmidt operators form itself a Hilbert space with inner product  $\langle A, B \rangle_{\mathcal{T}_2} \stackrel{\text{def}}{=} \text{Tr}A^*B$ .

In particular for any  $\mathbf{H} \in \mathcal{T}_1$  there holds by properties of the trace  $\text{Tr}(X\mathbf{H}) \leq \|\mathbf{H}\|_1\|X\|$ , where  $\|\cdot\|$  denotes the operator norm. Hence for  $X = \mathbf{S}_\mu(\alpha)$  given by (2) one can define with

analogy to ordinary Fourier transform [12], [13] a mapping  $\mathbf{F}^{(\alpha)} : \mathcal{T}_1 \rightarrow \mathcal{L}_2(\mathbb{R}^2)$  via

$$(\mathbf{F}^{(\alpha)}\mathbf{H})(\mu) \stackrel{\text{def}}{=} \text{Tr}(\mathbf{S}_\mu^*(\alpha)\mathbf{H}) = \langle \mathbf{S}_\mu(\alpha), \mathbf{H} \rangle_{\mathcal{T}_2} \quad (4)$$

Note that  $(\mathbf{F}^{(\alpha)}\mathbf{H})(0) = \text{Tr}\mathbf{H}$  and  $|(\mathbf{F}^{(\alpha)}\mathbf{H})(\mu)| \leq \|\mathbf{H}\|_1$ . The function  $\mathbf{F}^{(\alpha)}\mathbf{H} \in \mathcal{L}_2(\mathbb{R}^2)$  is sometimes called the "non-commutative" Fourier transform [11], inverse Weyl transform [14] or  $\alpha$ -generalized *spreading function* of  $\mathbf{H}$  [3].

**Lemma 1 (Spreading Representation)** *Let  $\mathbf{H} \in \mathcal{T}_2$ . Then there holds*

$$\mathbf{H} = \int (\mathbf{F}^{(\alpha)}\mathbf{H})(\mu) \mathbf{S}_\mu(\alpha) d\mu = \int \langle \mathbf{S}_\mu(\alpha), \mathbf{H} \rangle_{\mathcal{T}_2} \mathbf{S}_\mu(\alpha) d\mu \quad (5)$$

where the integral is meant in the weak sense.

The extension to  $\mathcal{T}_2$  is due to continuity of  $\mathbf{F}^{(\alpha)} : \mathcal{T}_1 \rightarrow \mathcal{L}_2(\mathbb{R}^2)$  and density of  $\mathcal{T}_1$  in  $\mathcal{T}_2$ . A complete proof of this lemma can be found for example in [11]. For  $\mathbf{H}, X \in \mathcal{T}_2$  the following Parseval-like identity

$$\langle X, \mathbf{H} \rangle_{\mathcal{T}_2} = \langle \mathbf{F}^{(\alpha)}X, \mathbf{F}^{(\alpha)}\mathbf{H} \rangle \quad (6)$$

holds. If we define the symplectic Fourier transform of a function  $\mathcal{F}_s : \mathbb{R}^2 \rightarrow \mathbb{C}$  as:

$$(\mathcal{F}_s F)(\mu) = \int_{\mathbb{R}^2} e^{-i2\pi\eta(\nu, \mu)} F(\nu) d\nu \quad (7)$$

then  $\mathcal{F}_s \mathbf{F}^{(\alpha)}$  establishes a correspondence between the ordinary function  $\mathbf{L}_X^{(\alpha)} = \mathcal{F}_s \mathbf{F}^{(\alpha)} X$  and an operator  $X$  (Weyl quantization [14]). The function  $\mathbf{L}_X^{(\alpha)}$  is called (generalized) *Weyl symbol* of  $X$ . The original Weyl symbol is  $\mathbf{L}_X^{(0)}$ . The cases  $\alpha = 1/2$  and  $\alpha = -1/2$  are also known as Kohn–Nirenberg symbol (or Zadeh's time-varying transfer function) and Bello's frequency-dependent modulation function [5]. Using Parseval identity for  $\mathcal{F}_s$  eq. (6) extends now to

$$\langle X, Y \rangle_{\mathcal{T}_2} = \langle \mathbf{F}^{(\alpha)}X, \mathbf{F}^{(\alpha)}Y \rangle = \langle \mathbf{L}_X^{(\alpha)}, \mathbf{L}_Y^{(\alpha)} \rangle \quad (8)$$

and consequentially  $\|X\|_2 = \|\mathbf{F}^{(\alpha)}X\|_2 = \|\mathbf{L}_X^{(\alpha)}\|_2$ .

## IV. EIGENSTRUCTURE OF OPERATORS WITH COMPACTLY SUPPORTED SPREADING

### A. The Approximate Eigenstructure

It is of general importance how much the Weyl symbol or a smoothed version of it approaches the eigenvalue characteristics of a given Hilbert–Schmidt operator. Since  $\mathbf{H} \in \mathcal{T}_2$  is compact, it has a Schmidt representation<sup>1</sup>  $\mathbf{H} = \sum_{n=1}^{\infty} s_n \langle x_k, \cdot \rangle y_k$  with the singular values  $\{s_k\}$  of  $\mathbf{H}$  and orthonormal bases  $\{x_k\}$  and  $\{y_k\}$ . However, the latter depends explicitly on  $\mathbf{H}$  and can be very unstructured. We are interested in a choice which is "more independent" of  $\mathbf{H}$ , which give rise to the following definition of what we will call "approximative" eigenstructure:

<sup>1</sup>For  $\mathbf{H}$  given in matrix representation also known as "Singular Value Decomposition"

**Definition 2 (Approximate Eigenstructure)** Let  $\lambda \in \mathbb{C}$ . If for  $g, \gamma : \mathbb{R} \rightarrow \mathbb{C}$  with  $\|g\|_p = \|\gamma\|_p = 1$  holds

$$\|\mathcal{H}\gamma - \lambda g\|_p \leq \epsilon(\lambda, \gamma, g) \quad (9)$$

we call  $\lambda$  an " $\ell_p$ -approximate eigenvalue" of  $\mathcal{H}$  with bound  $\epsilon(\lambda, \gamma, g)$ .

Obviously we have  $\epsilon(s_k, x_k, y_k) = 0$  for each  $p$ . The question is, how much particular choices for  $\lambda(\mu)$  and functions  $S_\mu \gamma$  and  $S_\mu g$  can approach the eigenstructure of a Hilbert Schmidt operator  $\mathcal{H}$ . Because in general these functions differ from the Schmidt representation of  $\mathcal{H}$  they will give an error which we have to control. Summarizing<sup>2</sup>:

$$E_p := \|\mathcal{H}S_\mu \gamma - \lambda(\mu)S_\mu g\|_p \leq \epsilon(\lambda(\mu), S_\mu \gamma, S_\mu g) \quad (10)$$

At this point we introduce furthermore the following abbreviation:  $\Sigma_{\mathcal{H}}^{(\alpha)} := \mathbf{F}^{(\alpha)} \mathcal{H}$  will always be the spreading function of  $\mathcal{H}$ . With  $U \subseteq \mathbb{R}^2$  we will denote its support and with  $|U| = \|\chi_U\|_1$  its size (measure).

### B. Previous Results

In [3, Theorem 5.6] W. Kozek has been considered the case  $\lambda = \mathbf{L}_{\mathcal{H}}^{(0)} = \mathcal{F}_s \Sigma_{\mathcal{H}}^{(0)}$  and  $g = \gamma$ . He obtained the following result:

**Theorem 3 (W. Kozek [3])** Let  $U = [-\tau_0, \tau_0] \times [-\nu_0, \nu_0]$ . If  $|U| = 4\tau_0\nu_0 \leq 1$  then

$$E_2^2 \leq 2 \sin\left(\frac{\pi|U|}{4}\right) \|\Sigma_{\mathcal{H}}^{(0)}\|_1^2 + \epsilon_\gamma \left( \|\Sigma_{\mathcal{H}^* \mathcal{H}}^{(0)}\|_1 + 2 \|\Sigma_{\mathcal{H}}^{(0)}\|_1^2 \right) \quad (11)$$

where  $\epsilon_\gamma = \|(\mathbf{A}_{\gamma\gamma}^{(0)} - 1)\chi_U\|_\infty$ .

Using  $\|\Sigma_{\mathcal{H}^* \mathcal{H}}^{(0)}\|_1 \leq \|\Sigma_{\mathcal{H}}^{(0)}\|_1^2$  eq. (11) can be written as

$$\frac{E_2^2}{\|\Sigma_{\mathcal{H}}^{(0)}\|_1^2} \leq 2 \sin\left(\frac{\pi|U|}{4}\right) + 3 \|(\mathbf{A}_{\gamma\gamma}^{(0)} - 1)\chi_U\|_\infty \quad (12)$$

G. Matz generalized the result of Theorem 3 in [4, Theorem 2.22] to a formulation in terms of weighted 1-moments of spreading functions which includes now different polarizations  $\alpha$  and is not restricted to the special choice of  $U$ . For  $U = [-\tau_0, \tau_0] \times [-\nu_0, \nu_0]$  and  $\alpha = 0$  the bounds agree with (12).

### C. New Related Results

Instead of directly considering the case of using the Weyl symbol for the approximation of the eigenstructure, we use a "smoothed" version:

$$\lambda = \mathcal{F}_s(\Sigma_{\mathcal{H}}^{(\alpha)} \cdot B) \quad (13)$$

and consider two cases:

**C1:** " $B = \mathbf{A}_{g\gamma}^{(\alpha)}$ ", such that  $\lambda = \mathbf{L}_{\mathcal{H}}^{(\alpha)} * \mathcal{F}_s \mathbf{A}_{g\gamma}^{(\alpha)}$  where  $*$  denotes convolution. This corresponds to the well-known smoothing with the cross Wigner function  $\mathcal{F}_s \mathbf{A}_{g\gamma}^{(\alpha)}$ .

<sup>2</sup>Note that  $S_\mu$  can be replaced with  $S_\mu(\beta)$  without change of  $E_p$

**C2:** " $B = 1$ ", such that  $\lambda(\mu) = \mathbf{L}_{\mathcal{H}}^{(\alpha)}(\mu)$ . This case is related to the symbol calculus and needed for comparisons with the previous results given so far.

Then the following theorem parallels Theorem 3 and its consequence (12).

**Theorem 4** For  $1 \leq p < \infty$  and  $1 \leq a \leq \infty$  holds:

$$\frac{E_p^p}{\|\Sigma_{\mathcal{H}}^{(\alpha)}\|_a^p} \leq \bar{\rho}_\infty^{p-2} \|(1 + |B|^2 - 2\operatorname{Re}\{\mathbf{A}_{g\gamma}^{(\alpha)} \bar{B}\})\chi_U\|_{b/p} \quad (14)$$

where  $1/a + 1/b = 1$ . The minimum over  $B$  is achieved for  $B = \mathbf{A}_{g\gamma}^{(\alpha)}$ .

*Proof:* The proof follows from the middle term of (23) in Lemma 6 given in the next section if one set  $W = \Sigma_{\mathcal{H}}^{(\alpha)}$  and  $K = \chi_U$ . The constant  $\bar{\rho}_\infty$  will also be explained later on. ■

It follows that for C2,  $p = 2$  and  $a = 1$  that

$$(14) \leq 2 \|(1 - \mathbf{A}_{g\gamma})\chi_U\|_\infty \quad (15)$$

which improves the previous bounds (11) and (12). It is independent of the polarization  $\alpha$  and does not require any shape or size constraints on  $U$ . Interestingly the offset in (12), which does not depend on  $(g, \gamma)$  and in a first attempt seems to be related to the notion of underspreadness, has been disappeared now.

## V. GENERALIZATION AND PROOFS

For the study of random operators we have to classify the overall spreading function. Thus we assume that all realizations of the spreading function can be written as

$$\Sigma_{\mathcal{H}}^{(\alpha)}(\mu) = K(\mu) \cdot W(\mu) \quad (16)$$

for a common function  $K : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ , which model some apriori knowledge (for example the square root of the scattering function in the WSSUS assumption [5] directly or some support knowledge). We will always denote with  $U \subseteq \mathbb{R}^2$  then the support of  $K$ . The function  $W : \mathbb{R}^2 \rightarrow \mathbb{C}$  represents the random part. From this considerations it is desirable to measure the error  $E_p$  with respect to a certain  $a$ -norm  $\|W\|_a$  of the random part, thus to look at the ratio  $E_p/\|W\|_a$ . We have the following Lemma:

**Lemma 5** Let  $\rho_p(\nu) := \|\mathbf{S}_\nu(\alpha)\gamma - B(\nu)g\|_p K(\nu)$ . For  $1 \leq p < \infty$ ,  $1 \leq a \leq \infty$  and  $1/a + 1/b = 1$  holds

$$E_p/\|W\|_a \leq \|\rho_p\|_b \quad (17)$$

whenever  $W \in \mathcal{L}_a(\mathbb{R}^2)$  and  $\rho_p \in \mathcal{L}_b(\mathbb{R}^2)$ .

*Proof:* Firstly – using Weyl's commutation rule and definition of  $\lambda$  in (13) gives us

$$\begin{aligned} E_p &\stackrel{(10)}{=} \left\| \int d\nu \Sigma_{\mathcal{H}}^{(\alpha)}(\nu) \mathbf{S}_\nu(\alpha) S_\mu \gamma - \lambda(\mu) S_\mu g \right\|_p \\ &\stackrel{(3)}{=} \left\| \mathbf{S}_\mu \left( \int d\nu \Sigma_{\mathcal{H}}^{(\alpha)}(\nu) e^{-i2\pi\eta(\nu, \mu)} \mathbf{S}_\nu(\alpha) \gamma - \lambda(\mu) g \right) \right\|_p \\ &\stackrel{(13)}{=} \left\| \int d\nu \Sigma_{\mathcal{H}}^{(\alpha)}(\nu) e^{-i2\pi\eta(\nu, \mu)} (\mathbf{S}_\nu(\alpha) \gamma - B(\nu)g) \right\|_p \end{aligned} \quad (18)$$

Note that  $p$ -norm is with respect to the argument of the functions  $g$  and  $\mathbf{S}_\nu(\alpha)\gamma$ . The last step follows because  $\mathbf{S}_\mu(\alpha)$  acts isometrically on all  $\mathcal{L}_p(\mathbb{R})$ . If we define

$$f(x, \nu) := e^{-i2\pi\eta(\nu, \mu)} \Sigma_{\mathcal{H}}^{(\alpha)}(\nu)[(\mathbf{S}_\nu(\alpha)\gamma)(x) - B(\nu)g(x)] \quad (19)$$

eq. (18) reads for  $1 \leq p < \infty$  by Minkowski (triangle) inequality

$$E_p = \left\| \int d\nu f(\cdot, \nu) \right\|_p \leq \left\| \int d\nu |f(\cdot, \nu)| \right\|_p \leq \int d\nu \|f(\cdot, \nu)\|_p \quad (20)$$

With  $\Sigma_{\mathcal{H}}^{(\alpha)} = W \cdot K$  and Hölder's inequality follows the claim of this lemma. ■

Let us fix for the moment  $\|\gamma\|_2 = \|g\|_2 = 1$  ( $\|\mathbf{A}_{g\gamma}^{(\alpha)}\|_\infty \leq 1$ ) and a constant  $C$  such that  $\max(\|\gamma\|_p, \|g\|_q) \leq C$ . Then it is obvious that for C1 and C2 follows  $\rho_p(\nu) \leq 2CK(\nu)$  and we have:

$$E_p/\|W\|_a \leq 2C\|K\|_b \quad (21)$$

In the next Lemma we will show that  $\|\rho_p\|_b$  can be related to weighted norms of ambiguity function, which we have studied already in [9]. A central role will play here the function:

$$\bar{\rho}(\nu) := \sup_x |(\mathbf{S}_\nu(\alpha)\gamma)(x) - B(\nu)g(x)|\chi_U(\nu) \quad (22)$$

which characterize the relative smoothness of  $g$  and  $\gamma$  with respect to shifts  $\nu \in U$ . Let us furthermore denote its supremum with  $\bar{\rho}_\infty := \|\bar{\rho}\|_\infty$ . Due to limited space we have to postpone a detailed discussion of  $\bar{\rho}$  and  $\bar{\rho}_\infty$  (which will be important for  $p \neq 2$  and then one has also to consider  $\bar{\rho}_\infty = \bar{\rho}_\infty(U)$ ) to a separate journal paper in preparation.

For simplicity we now make w.l.o.g. the assumption  $\|g\|_2 = \|\gamma\|_2 = 1$  and define the non-negative function  $R := 1 + |\mathbf{A}_{g\gamma}^{(\alpha)} - B|^2 - |\mathbf{A}_{g\gamma}^{(\alpha)}|^2$ .

**Lemma 6** *With the assumptions of Lemma 5 holds:*

$$\|\rho_p\|_b \leq \bar{\rho}_\infty^{\frac{p-2}{p}} \|RK^p\|_{b/p}^{1/p} \quad (23)$$

with equality for  $p = 2$ . The minimum over  $B$  of the rhs is achieved for C2.

*Proof:* We have for  $p \geq 1$ :

$$\begin{aligned} \rho_p(\nu) &\leq \bar{\rho}_\infty^{\frac{p-2}{p}}(\nu) \left( \int (|(\mathbf{S}_\nu(\alpha)\gamma)(x) - B(\nu)g(x)|^2 dx \right)^{1/p} K(\nu) \\ &= (\bar{\rho}(\nu)K(\nu))^{\frac{p-2}{p}} \rho_2(\nu)^{2/p} \end{aligned} \quad (24)$$

with equality for  $p = 2$  such that

$$\|\rho_p\|_b \leq \bar{\rho}_\infty^{\frac{p-2}{p}} \|\rho_2^2 K^{p-2}\|_{b/p}^{1/p} =: \bar{\rho}_\infty^{\frac{p-2}{p}} \|RK^p\|_{b/p}^{1/p} \quad (25)$$

From the definition of  $R$  it is obvious that the minimum of the bound in (24) is taken at  $B(U) = \mathbf{A}_{g\gamma}^{(\alpha)}(U)$  which is provided by C1. Because equality for  $p = 2$  in (24) this is also the optimizer for  $\|\rho_2\|_b$  for any  $b$ . ■

#### A. Relation to Weighted Ambiguity Norms

Now we will discuss the connection to weighted norms of ambiguity functions and fidelity criteria related to pulse shaping as introduced in [9]. It will give (partially) new insights into the terms of underspreadness in this context. Assume that  $R_\infty := \|R\chi_U\|_\infty \leq 1$  and  $b \geq p$ . Then it can be shown that

$$\|\rho_p\|_b \leq \bar{\rho}_\infty^{\frac{p-2}{p}} \left( \int R K^b \right)^{1/b} \quad (26)$$

If the latter can not be fulfilled, hence for  $b < p$  or if  $R_\infty > 1$ , it still holds:

$$\|\rho_p\|_b \leq \bar{\rho}_\infty^{\frac{p-2}{p}} R_\infty^{1/p} \|K\|_b \quad (27)$$

It can be verified that for C1 the condition  $R_\infty \leq 1$  is always fulfilled. Thus, in this case (26) reads:

$$\|\rho_p\|_b \leq \bar{\rho}_\infty^{\frac{p-2}{p}} \left( \|K^b\|_1 - \|(\mathbf{A}_{g\gamma}^{(\alpha)})^2 K^b\|_1 \right)^{1/b} \quad (28)$$

We conclude that with our assumptions a maximization of the "2-channel fidelity"  $\|(\mathbf{A}_{g\gamma}^{(\alpha)})^2 K^b\|_1$  (the case  $r = 2$  in [9]) controls  $E_p/\|W\|_a$ . This is also the term important for pulse shaping with respect to scattering function of WSSUS channels [3], [8].

But for the condition C2 the behavior is different. We arrive at the very interesting condition, that  $R_\infty \leq 1$  is equivalent to

$$\frac{1}{2} \leq \inf_{\mu \in U} \operatorname{Re}\{\mathbf{A}_{g\gamma}^{(\alpha)}(\mu)\} \quad (29)$$

We will show later on that this condition can not be fulfilled on every  $U$ . However, if (29) holds we get from (26) and (29):

$$\|\rho_p\|_b \leq \bar{\rho}_\infty^{\frac{p-2}{p}} \left( 2(\|K^b\|_1 - \|\operatorname{Re}\{\mathbf{A}_{g\gamma}^{(\alpha)}\} K^b\|_1) \right)^{1/b} \quad (30)$$

which is then a problem of the maximization of the "1-channel fidelity" (the case  $r = 1$  in [9]).

#### VI. SOME NECESSARY SUPPORT CONDITIONS

In this section we present some necessary condition on  $|U|$  for the case  $K = \chi_U$  (therefore support conditions on  $\Sigma_{\mathcal{H}}^{(\alpha)}$ ) which follow from the methods presented in [9]. Due to limited space we have to omit the proofs, which will then appear separately in a journal version. In particular one can show from [9] that for  $|U| \leq e \min(1, 2/r)$  follows

$$\|(\mathbf{A}_{g\gamma}^{(\alpha)})^r \chi_U\|_1 \leq |U| e^{-\frac{|U|r}{2e}} \quad (31)$$

and with this result follows:

**Lemma 7 (Necessary Condition for C2)** *The condition in (29) can only be fulfilled if  $|U| \leq 2e \ln 2$ .*

We believe that this bound is very coarse and it should be possible to improve it in using more advanced techniques. Further support results follow for  $b \geq p$ . We define from (28) the following quantity

$$r_1(U) := \bar{\rho}_\infty^{\frac{p-2}{p}} \left( |U| - \|(\mathbf{A}_{g\gamma}^{(\alpha)})^2 \chi_U\|_1 \right)^{1/b} \quad (32)$$

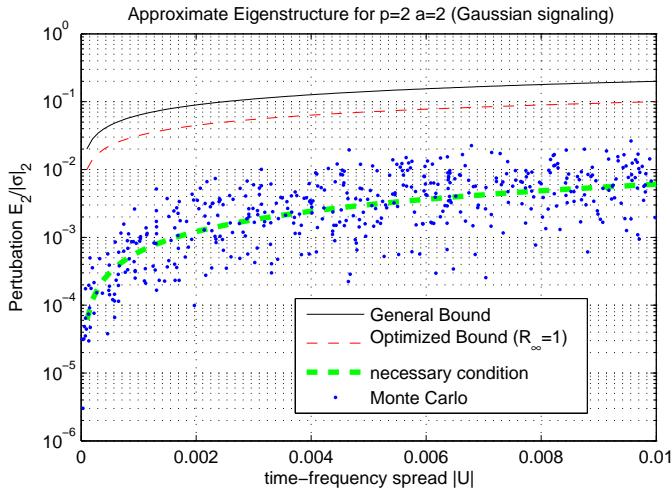


Fig. 1. Approximation error  $E_2/\|\Sigma_{\mathcal{H}}^{(\alpha)}\|_2$  for the case C2. "General bound" refers to (21), "Optimized Bound ( $R_\infty = 1$ )" is (27) and "necessary condition" is the results of (34). The monte carlo data is obtained from Gaussian signaling and a rectangular spreading function with independent complex normal distributed components  $c_k$  ( $K = 10$ ) as explained in (35).

for the case C1. For C2 we have to guarantee that  $R_\infty \leq 1$  and we define instead from (30) the quantity:

$$r_2(U) := \bar{\rho}_\infty^{\frac{p-2}{p}} \left( 2(|U| - \|\text{Re}\{\mathbf{A}_{g\gamma}^{(\alpha)}\} \chi_U\|_1) \right)^{1/b} \quad (33)$$

Then the following can be shown:

**Lemma 8 (Necessary Conditions on  $|U|$ )** Let  $1 > \delta_k > 0$ ,  $|U| \leq e$  and  $k = \{1, 2\}$ . If  $r_k(U) \leq \delta_k$  then  $U$  has to fulfill:

$$\bar{\rho}_\infty^{\frac{p-2}{p}} \left( k|U|(1 - e^{-\frac{|U|}{ek}}) \right)^{1/b} \leq \delta_k \quad (34)$$

The latter is an implicit inequality for  $|U|$ . However, it is possible to obtain from this an explicit upper bound for  $|U|$  for a given  $\delta_k$  (not shown in the paper).

## VII. NUMERICAL VERIFICATION

In the following we will evaluate and test the obtained bounds for Gaussian signaling, i.e.  $g$  and  $\gamma$  are time–frequency symmetric Gaussian functions. We consider a spreading function given as:

$$\Sigma(\nu) = \sum_{k \in \mathbb{Z}_K^2} c_k \chi_Q(\nu - u(k + o)) \quad (35)$$

where  $\mathbb{Z}_K = \{0 \dots K-1\}$ ,  $Q = [0, u] \times [0, u]$  and  $o = (\frac{1}{2}, \frac{1}{2})$ . If we fix the support of the spreading function to be  $|U|$ , then follows  $u = \sqrt{|U|}/K$ . For such a model the  $a$ -norm of the spreading function is:  $\|\Sigma\|_a = u^{2/a} \|c\|_a$ , where  $\|c\|_a$  is simply the  $a$ th vector norm of the vector  $c$  with coefficients  $c_k$ . The error  $E_p$  can be simplified much for Gaussian signaling and finally computed numerically (therefore Gaussian signaling was chosen). Fig. 1 shows the results of several monte carlo runs, each corresponds to one point in the plot. For comparison the various bounds and results are included

in the plot (more details in the caption). Clearly the most important result ("necessary condition") can not serve as a bound. However, its interesting that it produce a rough value of the approximation error.

## VIII. CONCLUSIONS

In this contribution we established in a more general fashion the problem of approximate eigenstructure of LTV channels. We extracted several criteria related to signaling in those channels and pulse shaping. We hope that our results give some more implications to the role of underspreadness for wireless communication. Furthermore the connection to symbol calculus of pseudo-differential operators is straightforward, such that insights into topics like approximate commutativity — or more generally speaking — approximations to spectral properties have to be expected.

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